数理解析研究所講究録
985 巻 1997 年 35-41

On Semicontinuity of Marginal Functions $\sup_{y\in F(x)} f(y)$ and $\inf_{y\in F(x)} f(y)$

新潟大学大学院 自然科学研究科 木村 寛 (YUTAKA KIMURA)*
新潟大学 理学部 田中 謙輔 (KENSUKE TANAKA)†
弘前大学 理学部 田中 環 (TAMAKI TANAKA)‡

1. Introduction

集合値写像の半連続性に対する研究や, それを用いた研究は, 長年多くの研究者によりなされてきている. 最近では特に, Luc [6] や, Tanaka and Seino [11] がベクトル値関数や, 集合値写像における半連続性について興味深い結果を与えており, また, Ferro [4] や, Tan, Yu, and Yuan [9] はこれらの概念をもとに集合値写像に対する最適化問題を論じている. このように, 集合値写像における最適化問題を論じる上で, 集合値写像の半連続性の概念はとても重要であると考えられる. よって我々は, 集合値写像における古典的な上, 下半連続性の一般化である cone-semicontinuity をいくつか定義し, そのような連続性に対して maximum theorem を体系づけることを目的とする. そこで, はじめにいくつかの cone-semicontinuity の関係を特徴づけ, その後, 実数値関数と集合値写像の合成写像の cone-semicontinuity について考察する. そして最後に, これらの結果をもとに 2 つのタイプの marginal function (*i.e.*, $\sup_{y\in F(x)} f(y)$, $\inf_{y\in F(x)} f(y)$) の半連続性に対する結果を与える.

2. Preliminaries

X を位相空間, Y を Y での凸錐 C で順序づけされた線形位相空間とする. ここで, 以下簡単のため凸錐 C は pointed (i.e., $C\cap(-C)=\{\theta_Y\}$) であると仮定し, また $\mathbf{int}\, C$ は空集合でないとする. ただし, θ_Y は Y での null vector とし, $\mathbf{int}\, C$ は C すべての内点の集合である. また, 記号 $\mathbf{cl}\, C$ とは, C の閉包を表す. Y の任意のベクトル y から空でない Y の部分集合 A への距離関数 $d_Y : Y \to \boldsymbol{R}$ を $d_Y(y, A) = \inf_{a\in A} d(y, a)$ で定義する.

F が X から Y への集合値写像であるとは, X から Y のべき集合 2^Y への写像であり, 記号 $F : X \rightsquigarrow Y$ で表すとする. 集合値写像 $F : X \rightsquigarrow Y$ に対して, $\mathbf{Graph}(F)$ は次の

---

*Department of Mathematical Science, Graduate School of Science and Technology, Niigata University, Niigata 950–21, JAPAN

†Department of Mathematics, Faculty of Science, Niigata University, Niigata 950–21, JAPAN

‡Department of Information Science, Faculty of Science, Hirosaki University, Hirosaki 036, JAPAN

(2.1) で定義される.

$$\mathbf{Graph}(F) := \{(x,y) \in X \times Y | y \in F(x)\}. \tag{2.1}$$

また, F の定義域とは $F(x)$ が空でない $x \in X$ 全体の集合, つまり,

$$\mathbf{Dom}F := \{x \in X | F(x) \neq \emptyset\} \tag{2.2}$$

であり, F の値域は

$$\mathbf{Im}F := \bigcup_{x \in X} F(x) \tag{2.3}$$

で定義される.

X と Y を線形位相空間とし, F を X から Y への集合値写像としたとき, F が x_0 で equally weak upper semicontinuous (**ewusc** for short) [11] であるとは, $\theta_Y \in Y$ での任意の開近傍 G に対して, x_0 での近傍 U が存在して,

$$F(x) \subset F(x_0) + G \quad \text{for all} \quad x \in U \cap \mathbf{Dom}F, \tag{2.4}$$

が成り立つことである. また, F が x_0 で equally lower semicontinuous (**elsc** for short) [11] であるとは, $\theta_Y \in Y$ での任意の開近傍 G に対して, x_0 での近傍 U が存在して,

$$F(x_0) \subset F(x) + G \quad \text{for all} \quad x \in U \cap \mathbf{Dom}F, \tag{2.5}$$

が成り立つことをいう.

次に, Marginal function に関する定理を挙げる. これは Maximum theorem [1, Th.1.4.16] と呼ばれ詳細は [1, Chapter1] で述べられている.

Proposition 1. Let X and Y be metric spaces, respectively. For a set-valued map $F : X \rightsquigarrow Y$ and a real-valued function $f : \mathbf{Graph}(F) \to \boldsymbol{R}$, we have the following statements.

(i) If f and F are lower semicontinuous in the sense of each definition so is the marginal function g is also a lower semicontinuous function.

(ii) If f and F are upper semicontinuous in the sense of each definition and if $F(x)$ is a compact set for each $x \in X$, the marginal function g is also an upper semicontinuous function.

3. Cone-Semicontinuity for Set-Valued Maps

ここでは, 集合値写像 $F : X \rightsquigarrow Y$ に対する cone-semicontinuity を定義し, 更にそれらの関係について論じていくが, はじめに集合値写像の古典的な upper semicontinuity の定義を挙げ, 次に, upper semicontinuity の拡張である cone-upper semicontinuity を定義する.

Definition 1. Let X and Y be topological spaces, respectively. A set-valued map $F : X \rightsquigarrow Y$ is said to be upper semicontinuous (**u.s.c.** for short) at x_0 if for any open set V with $F(x_0) \subset V$, there exists a neighborhood U of x_0 such that

$$F(x) \subset V \quad \text{for all} \quad x \in U. \tag{3.1}$$

Definition 2. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. A set-valued map $F : X \rightsquigarrow Y$ is said to be:

(u1) C-upper semicontinuous at x_0 (C-**usc**) if for any open neighborhood V of $F(x_0)$, there exists an open neighborhood U of x_0 such that $F(x) \subset V + C$ for all $x \in U \cap \mathbf{Dom}F$ ([6, Def.7.1(p.33)]);

(u2) C-weak upper semicontinuous at x_0 (C-**wusc**) if for any open neighborhood V of **cl** $F(x_0)$, there exists an open neighborhood U of x_0 such that $F(x) \subset V + C$ for all $x \in U \cap \mathbf{Dom}F$;

(u3) C-equally weak upper semicontinuous at x_0 (C-**ewusc**) if for any open neighborhood G of $\theta_Y \in Y$, there exists an open neighborhood U of x_0 such that $F(x) \subset F(x_0) + G + C$ for all $x \in U \cap \mathbf{Dom}F$.

上述の 3 つの集合値写像における cone-upper semicontinuities は, [11] で述べられている集合値写像の upper semicontinuity や, weak upper semicontinuity, equally upper semicontinuity の一般化である. もちろん, 古典的な upper semicontinuity であれば cone-upper semicontinuity であり, また cone-upper semicontinuity は実数値関数の一般の下半連続性や, また, ベクトル値関数の下半連続性の拡張になっている [10, Def.2.1].

Remark 1. In [4], Ferro denote condition **(u1)** above the terminology "upper C-continuity ". When $C = \{\theta_Y\}$ in Definition 2., a set-valued map $F : X \rightsquigarrow Y$ is C-usc at x_0 if and only if F is u.s.c. at x_0.

Proposition 1. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. A set-valued map $F : X \rightsquigarrow Y$ satisfies the condition **(u3)** at x_0 if and only if F satisfies the following condition:

(u3)' For any $d \in \mathbf{int}\ C$, there exists an open neighborhood U of x_0 such that $F(x) \subset F(x_0) - d + \mathbf{int}\ C$ for all $x \in U$.

(u1), **(u2)**, そして **(u3)** の関係について次の Proposition 2. が成立する.

Proposition 2. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. In the above definition, we have **(u1)** $\Rightarrow$ **(u2)** $\Rightarrow$ **(u3)**.

Example 1. (**(u2)** であるが **(u1)** ではない例) Let $X = Y = \boldsymbol{R}$ and $C = \boldsymbol{R}_+$. We consider the following set-valued map F from $\boldsymbol{R}$ to $\boldsymbol{R}$ defined by

$$F(x) = \left\{ y \in \boldsymbol{R} \middle| -x^2 < y \leq 1 \right\}. \tag{3.2}$$

We can verify that F is $\boldsymbol{R}_+$-wusc at $x = 0$ but not $\boldsymbol{R}_+$-usc at the point, where $\boldsymbol{R}_+ = \{r \in \boldsymbol{R} | r \geq 0\}$.

Example 2. (**(u3)** であるが **(u2)** ではない例) Let $X = \boldsymbol{R}_+$, $Y = \boldsymbol{R}^2$ and $C = \boldsymbol{R}^2_+$. We consider the following set-valued map F from $\boldsymbol{R}$ to $\boldsymbol{R}$ defined by

$$F(x) = \left\{ (z_1, z_2) \in \boldsymbol{R}^2 \;\middle|\; z_2 > \frac{1}{z_1 + x}, z_1 \geq 0 \right\}. \tag{3.3}$$

We can verify that F is $\boldsymbol{R}^2_+$-ewusc at $x = 0$ but not $\boldsymbol{R}^2_+$-wusc at the point.

Proposition 3. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. A set-valued map $F : X \rightsquigarrow Y$ satisfies the condition **(u1)** x_0 if and only if F satisfies the following condition:

(u1)' For any open set V with $F(x_0) \subset V + C$, there exists an open neighborhood U of x_0 such that $F(x) \subset V + C$ for all $x \in U \cap \mathbf{Dom}F$;

Also, a set-valued map $F : X \rightsquigarrow Y$ satisfies the condition **(u2)** at x_0 if and only if F satisfies the following condition:

(u2)' For any open set V with **cl** $F(x_0) \subset V + C$, there exists an open neighborhood U of x_0 such that $F(x) \subset V + C$ for all $x \in U \cap \mathbf{Dom}F$;

Proposition 4. Let X and Y be a topological space and an ordered metric and vector space with a convex cone C, respectively, where the metric of Y is denoted by d_Y. A set-valued map $F : X \rightsquigarrow Y$ satisfies the condition **(u3)** at x_0 if and only if F satisfies the following condition:

(u3)" For any $\varepsilon > 0$, there exists an open neighborhood U of x_0 such that

$$F(x) \subset B_Y(F(x_0), \varepsilon) + C, \qquad \forall x \in U \cap \mathbf{Dom}F,$$

where $B_Y(A, \varepsilon) := \{y \in Y \mid d_Y(y, A) < \varepsilon\}$.

Proposition 5. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. In the above definition, if $F(x_0)$ is closed then **(u2)** $\Rightarrow$ **(u1)**. Also, **cl** $F(x_0)$ is compact in Y, then **(u3)** $\Rightarrow$ **(u2)**.

次に, 集合値写像に対する古典的な lower semicontinuity の定義を挙げ, 更に cone-lower semicontinuity を次に定義する.

Definition 3. Let X and Y be topological spaces. A set-valued map $F : X \rightsquigarrow Y$ is said to be lower semicontinuous (**l.s.c.** for short) at x_0 if for any open set V with $F(x_0) \cap V \neq \emptyset$, there exists an open neighborhood U of x_0 such that

$$F(x) \cap V \neq \emptyset \quad \text{for all} \quad x \in U. \tag{3.4}$$

Definition 4. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. A set-valued map $F : X \rightsquigarrow Y$ is said to be:

(l1) C-equally lower semicontinuous at x_0 (C-**elsc**) if for any neighborhood G of $\theta_Y \in Y$, there exists a neighborhood U of x_0 such that $F(x_0) \subset F(x) + G - C$ for all $x \in U \cap \mathbf{Dom}F$;

(l2) C-lower semicontinuous at x_0 (C-**lsc**) if for any $y_0 \in F(x_0)$ and any neighborhood G of $\theta_Y \in Y$, there exists a neighborhood U of x_0 with $F(x) \cap (y_0 + G + C) \neq \emptyset$ for any $x \in U \cap \mathbf{Dom}F$.

この２つの集合値写像における cone-lower semicontinuities は, [11] で述べられている集合値写像の equally lower semicontinuity や, lower semicontinuity の一般化である. もちろん, 古典的な lower semicontinuity であれば, C-lower semicontinuity である.

Remark 2. In [4], Ferro denote condition **(l1)** above by the terminology "lower C-semicontinuity". When $C = \{\theta_Y\}$ in Definition def-C-lsc, a set-valued map $F : X \rightsquigarrow Y$ is C-lsc at x_0 if and only if F is l.s.c. at x_0.

Proposition 6. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. In the above definition, **(l1)** $\Rightarrow$ **(l2)**. If **cl** $F(x_0)$ is compact, the converse is true. See Ferro [4] in detail.

4. Cone-Semicontinuity of Composite Maps and Marginal Functions

実数値関数 $f : Y \to \boldsymbol{R}$ が $x_0 \in Y$ で upper semicontinuous (u.s.c. for short) であるとは, 任意の正の実数 $\varepsilon > 0$ に対して, x_0 での近傍 U が存在し, すべての $x \in U$ で $f(x) - f(x_0) < \varepsilon$ が成り立つことをいう. f が Y 上で u.s.c. であるための必要十分条件は, 任意の実数 $a \in \boldsymbol{R}$ に対して, Y での部分集合 $\{ x \in Y \mid f(x) < a \}$ が開集合となることである. また, $-f$ が x_0 で u.s.c. であるとき f は x_0 で lower semicontinuous (l.s.c. for short) という.

次の Theorem 1. を示すために, 実数値関数における上記の upper semicontinuity や lower semicontinuity よりも強い概念を導入する.

Definition 1. Let Y be a topological vector space. A real-valued function $f : Y \to \boldsymbol{R}$ is called monotonically u.s.c. (resp., monotonically l.s.c.) if for any $\varepsilon > 0$, there exists a neighborhood G of $\theta_Y \in Y$ such that $f^{-1}(V + (-\varepsilon, \varepsilon) + \boldsymbol{R}_-)$ is open and $f^{-1}(V) + G \subset f^{-1}(V + (-\varepsilon, \varepsilon) + \boldsymbol{R}_-)$ for all $V \subset \boldsymbol{R}$ (resp., by replacing $\boldsymbol{R}_+$ by $\boldsymbol{R}_-$, where $\boldsymbol{R}_- = \{r \in \boldsymbol{R} | r \leq 0\}$).

実数値関数と集合値関数の合成写像 $\varphi : \mathbf{Dom}F \rightsquigarrow \boldsymbol{R}$ を以下で定義する.

$$\varphi(x) := f \circ F(x) = \bigcup_{y \in F(x)} \{f(y)\}. \tag{4.1}$$

また, 以後凸錐 C は $C = \boldsymbol{R}_+$ または $C = \boldsymbol{R}_-$ で考える.

Theorem 1. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. For $F : X \rightsquigarrow Y$ with $\mathbf{Dom}F \neq \emptyset$ and $f : Y \to \boldsymbol{R}$, we have the following:

(1a) if F is u.s.c. and f is u.s.c. then φ is $\boldsymbol{R}_-$-ewusc;

(1b) if F is ewusc and f is monotonically u.s.c. then φ is $\boldsymbol{R}_-$-ewusc;

(2a) if F is u.s.c. and f is l.s.c. then φ is $\boldsymbol{R}_+$-ewusc;

(2b) if F is ewusc and f is monotonically l.s.c. then φ is $\boldsymbol{R}_+$-ewusc;

(3) if F is elsc and f is monotonically u.s.c. then φ is $\boldsymbol{R}_+$-elsc;

(4) if F is elsc and f is monotonically l.s.c. then φ is $\boldsymbol{R}_-$-elsc;

(5) if F is l.s.c. and f is u.s.c. then φ is $\boldsymbol{R}_-$-lsc;

(6) if F is l.s.c. and f is l.s.c. then φ is $\boldsymbol{R}_+$-elsc.

ここで, 次の２つのタイプの marginal function を定義する.

$$\sup \varphi(x) := \sup_{y \in F(x)} f(y), \tag{4.2}$$

$$\inf \varphi(x) := \inf_{y \in F(x)} f(y), \tag{4.3}$$

ただし, $F : X \rightsquigarrow Y$ は集合値写像であり $f : Y \to \boldsymbol{R}$ は実数値関数である.

Lemma 1. Let X be a topological space For a set-valued map $\varphi : X \rightsquigarrow \boldsymbol{R}$ is $\boldsymbol{R}_-$-ewusc (resp. $\boldsymbol{R}_-$-elsc, $\boldsymbol{R}_-$-lsc) if and only if $-\varphi$ is $\boldsymbol{R}_+$-ewusc (resp. $\boldsymbol{R}_+$-elsc, $\boldsymbol{R}_+$-lsc).

Theorem 2. Let X be a topological space. For a set-valued map $\varphi : X \rightsquigarrow \boldsymbol{R}$, we have the following:

(1) if φ is $\boldsymbol{R}_-$-ewusc then $\sup \varphi$ is u.s.c.;

(2) if φ is $\boldsymbol{R}_+$-ewusc then $\inf \varphi$ is l.s.c.;

(3) if φ is $\boldsymbol{R}_+$-elsc then $\sup \varphi$ is l.s.c.;

(4) if φ is $\boldsymbol{R}_-$-elsc then $\inf \varphi$ is u.s.c.;

(5) if φ is $\boldsymbol{R}_-$-lsc then $\inf \varphi$ is u.s.c.;

(6) if φ is $\boldsymbol{R}_+$-lsc then $\sup \varphi$ is l.s.c..

Theorem 1., Theorem 2. から次の Corollary 1. が得られる.

Corollary 1. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. Let $F : X \rightsquigarrow Y$ be a set-valued map with $\mathbf{Dom}F \neq \emptyset$ and $f : Y \to \boldsymbol{R}$. For the marginal function is defined by (4.2) and (4.3), we have the following:

(1a) if F is u.s.c. and f is u.s.c. then $\sup \varphi$ is u.s.c.;

(1b) if F is ewusc and f is monotonically u.s.c. then $\sup \varphi$ is u.s.c.;

(2a) if F is u.s.c. and f is l.s.c. then $\inf \varphi$ is l.s.c.;

(2b) if F is ewusc and f is monotonically l.s.c. then $\inf \varphi$ is l.s.c.;

(3) if F is elsc and f is monotonically u.s.c. then $\sup \varphi$ is l.s.c.;

(4) if F is elsc and f is monotonically l.s.c. then $\inf \varphi$ is u.s.c.;

(5) if F is l.s.c. and f is u.s.c. then $\inf \varphi$ is u.s.c.;

(6) if F is l.s.c. and f is l.s.c. then $\sup \varphi$ is l.s.c..

References

[1] J. -P. AUBIN and H. FRANKOWSKA, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[2] G. BEER, *Topologies on Closed and Closed Convex Sets*, Kluwe Academic Publishers, Dordrecht, 1993.

[3] C. BERGE, *Topological Spaces*, Macmillan, New York, 1963.

[4] F. FERRO, *An Optimization Result for Set-Valued Mappings and a Stability Property in Vector Problems with Constraints*, JOTA, Vol.90, No.1, pp.63–77, 1996.

[5] W. W. HOGAN, *Point-to-Set Maps in Mathematical Programming*, SIAM Review, Vol.15, pp.591–603, 1973.

[6] D. T. LUC, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, Vol.319, Springer-Verlag, Berlin, 1989.

[7] D. G. LUENBERGER, *Optimization by Vector Space Methods*, John Wiley & Sons, 1969.

[8] R. T. ROCKAFELLAR, *Convex Analysis*, Princeton University Press, 1970.

[9] K.K. TAN, J. YU, and X.Z. YUAN, *Existence Theorems for Saddle Points of Vector-Valued Maps*, JOTA, Vol.89, pp.731–747, 1996.

[10] T. TANAKA, *Generalized Semicontinuity and Existence Theorems for Cone Saddle Points*, to appear in Journal of Applied Mathematics and Optimization, 1996.

[11] T. TANAKA and T.SEINO, *On a Theoretically Conformable Duality for Semicontinuity of Set-Valued Mappings,*, Vol.939, pp.1–10, RIMS koukyuroku, Vol.939, pp.1–10, 1996.